Claim: $\{a_n\}_n$ is an arith. seq. $(a_n = a_0 + n \cdot d)$

$\iff \exists d \, \forall n \quad a_{n+1} - a_n = d \quad (*)$

proof:

$(\Rightarrow)$ $\quad a_{n+1} = a_0 + (n+1) \cdot d$
$\quad a_n = a_0 + n \cdot d$ $\quad \Rightarrow \quad a_{n+1} - a_n = d$

$(\Leftarrow)$ by ind. on $n$.

basis: $n = 0$: $\quad a_1 - a_0 = d \quad \Rightarrow \quad a_1 = a_0 + 1 \cdot d$

hyp: $a_n = a_0 + n \cdot d$

step: $a_{n+1} \underset{\text{by } (*)}{=} a_n + d \underset{\text{ind. hyp.}}{=} (a_0 + n \cdot d) + d = a_0 + (n+1) \cdot d$ $\boxtimes$

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Claim: if $a_n \hat{=} a_0 + n \cdot d$, $S_n \hat{=} \sum_{i=0}^{n} a_i$

then $S_n = a_0 (n+1) + d \cdot \frac{n(n+1)}{2}$

proof: $\hat{S}_n = \sum_{i=0}^{n} i$ satisfies $\hat{S}_n = \frac{1}{2} n(n+1)$

$$S_n \hat{=} \sum_{i=0}^{n} a_i = \sum_{i=0}^{n} (a_0 + i \cdot d)$$

$$= a_0 \cdot (n+1) + d \cdot \sum_{i=0}^{n} i$$

$$= a_0 \cdot (n+1) + d \cdot \frac{n \cdot (n+1)}{2}$$ $\boxtimes$

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Lemma: $q^{n+1} - 1 = (q-1)(1 + q + \dots + q^n)$

proof: $(q-1) \cdot (1 + q + \dots + q^n)$

$$= \begin{array}{l} q + q^2 + q^3 + \dots + q^n + q^{n+1} \\ -1 - q - q^2 - q^3 - \quad - q^n \end{array}$$

$$= \begin{array}{l} \cancel{q} + \cancel{q^2} + \cancel{q^3} + \dots + \cancel{q^n} + q^{n+1} \\ -1 - \cancel{q} - \cancel{q^2} - \cancel{q^3} - \quad - \cancel{q^n} \end{array}$$

$$= -1 + q^{n+1}$$ $\boxtimes$

Corollary: $b_n \hat{=} b_0 \cdot q^n$, $S_n = \sum_{i=0}^{n} b_i$, $q \neq 1$

$$\Rightarrow S_n = b_0 \cdot \left( \frac{q^{n+1} - 1}{q - 1} \right)$$

proof:

$$S_n = b_0 (1 + q + \dots + q^n)$$

$$= b_0 \cdot \frac{q^{n+1} - 1}{q - 1}$$ $\boxtimes$

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Theorem: $H_{2^k} \leq 1 + k$

proof: by ind. on $k$.

basis: $k = 0$: $\quad H_{2^0} = H_1 = 1 \leq 1 + 0$

hyp: $H_{2^k} \leq 1 + k$

step:

$$H_{2^{(k+1)}} = H_{2^k} + \sum_{i=2^k+1}^{2 \cdot 2^k} \frac{1}{i}$$

$$\leq 1 + k + \sum_{i=2^k+1}^{2 \cdot 2^k} \frac{1}{2^k} \quad (\text{ind hyp} + i > 2^k)$$

$$= 1 + k + 2^k \cdot \frac{1}{2^k} = 1 + (k+1)$$ $\boxtimes$

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Theorem: $1 + \frac{k}{2} \leq H_{2^k}$

proof: same proof with $i \leq 2 \cdot 2^k$,

hence $\frac{1}{i} \geq \frac{1}{2 \cdot 2^k}$ ....