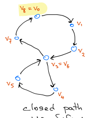
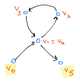
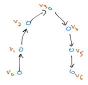


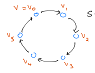
closed path
מסלול סגור



open path
מסלול פתוח

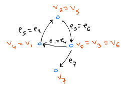


simple path
מסלול פשוט



simple cycle
מעגל פשוט

self-loop
לולאה עצמית



open path
not simple

Directed Acyclic Graph
גרף מכוון חסר מעגלים

Lemma: if p is not a simple path, then $\exists q \subseteq p$ s.t. q is a simple cycle

proof: seq of vertices in p is $\{v_i\}_{i=0}^{\ell}$.
let $i<j$ be a pair such that:
1) $v_i = v_j$
2) if $v_k = v_\ell$ for $k<\ell$, then $j-i \le \ell-k$.

[(i,j) is the "closest" pair of "repeating" vertices in p]

now: $v_i, \dots, v_{j-1}$ are distinct (why?)
so $v_i \to v_{i+1} \to \dots \to v_j$ is a simple cycle. ☒

Lemma: Let G denote a DAG.
$\forall$path p in G: $\text{length}(p) \le n-1$.

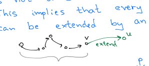
proof: every path p in G is simple.
(otherwise, $\exists q \subseteq p$ which is a cycle, contradicting G is acyclic).

Let $\ell \triangleq \text{length}(p)$

Let $\{v_i\}_{i=0}^{\ell}$ denote seq. of vertices in p. By the pigeonhole principle, $\ell+1 \le n$. ☒

Lemma: $\forall$DAG G $\exists v$ sink

proof: assume, for the sake of contradiction, that every $v \in V$ is not a sink.
This implies that every path p can be extended by an edge:



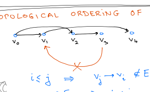
Hence, G contains a path p of length $\ge |V|$, this contradicts the prev. lemma. ☒

Lemma: $\forall$DAG G $\exists v$ source

proof: could use a "path extension argument" as before (how?).
Alternatively, define reverse(G):
$G = (V,E)$ & reverse$(G) = (V, \text{reverse}(E))$
$x \to y \in E \iff y \to x \in \text{reverse}(E)$
1) G DAG $\iff$ reverse(G) DAG
2) v source in G $\iff$ v sink in reverse(G)

By prev. lemma $\exists v$ sink in reverse(G).
$\Rightarrow \exists v$ source in G ☒

TOPOLOGICAL ORDERING OF A DAG

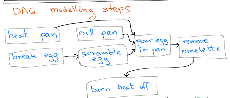


NOTE:
$v_i \to v_i \notin E$
because a DAG has no self loops

$i \le j \Rightarrow v_j \to v_i \notin E$

contra-positive
$v_j \to v_i \in E \Rightarrow j < i$

DAG modelling steps

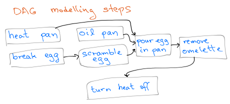


If task performed by a single person, how should person order subtasks?
Topological ordering!

Good
* "oil pan" before "pouring egg"
* "heat pan" before/after "break egg".

Bad
* "pouring egg" before "oil pan"

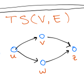
DAG modelling steps



Q: can 2 people prepare an omelette faster?
What about 3 people? 4? 5? ...
("parallelism")

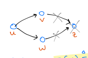
TS(V, E)

$G=(V,E)$



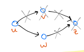
$V = \{u, v, w, z\}$
$E = \{u \to v,\ u \to w,\ v \to z,\ w \to z\}$

z is a sink: $\pi(z) \triangleq 3$



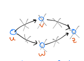
$V \setminus \{z\} = \{u, v, w, \cancel{z}\}$
$E \setminus E_z = \{u \to v, u \to w\}$

v is a sink: $\pi(v) \triangleq 2$



$V \setminus \{v, z\} = \{u, \cancel{v}, w, \cancel{z}\}$
$E \setminus (E_v \cup E_z) = \{u \to w\}$

w is a sink: $\pi(w) \triangleq 1$

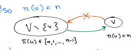


$V \setminus \{w, v, z\} = \{u, \cancel{v}, \cancel{w}, \cancel{z}\}$
$E \setminus (E_w \cup E_v \cup E_z) = \emptyset$

only u left (stopping cond.):
$\pi(u) \triangleq 0$

CORRECTNESS OF ALG TS(V, E):
* $G=(V,E)$ DAG $\Rightarrow \pi$ is topo. ord.

proof: ind. on $|V|$.
basis: $|V| = 1$. let $v \in V$, then $\pi(v)=0$.
hyp: (*) holds if $|V| = n$.
step: need to prove (*) for $|V| = n+1$.
alg picks sink v (always exists, why?).
Now: ind. hyp: $TS(V \setminus \{v\}, E \setminus E_v)$
finds $\pi : V \setminus \{v\} \to \{0, \dots, n-1\}$ topo. ord.
also $\pi(v) = n$



claim: π is a topo. ord.
consider $x \to y \in E$.
if $x, y \in V \setminus \{v\}$, then $\pi(x) < \pi(y)$ (why?)
if $x = v$, then v not sink (contradict.)
if $y = v$ then $\pi(v) = n > \pi(x)$ ✓ ☒