THM: $f: A \xrightarrow{1-1} B \quad (|A|, |B| < \infty)$

$\Rightarrow \quad |A| \leq |B|$

proof: ind. on $|A|$.

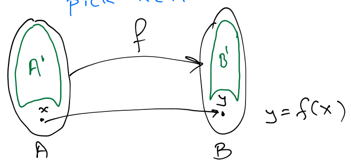
basis: $|A| = 0$. then clearly $0 \leq |B|$.

(what is $f: \emptyset \to B$?)

hyp: THM holds for $|A| = n$

step: prove THM for $|A| = n+1$.

pick $x \in A$



$A' \triangleq A \setminus \{x\}$

$B' \triangleq B \setminus \{f(x)\}$

$g \triangleq$ restriction of f to domain A':

$g: A' \to B'$ (stronger than B)

$(\forall z \in A': \ g(z) = f(z) \neq f(x).$

$\Rightarrow \text{range}(g) \subseteq B \setminus \{f(x)\} = B')$

now, g is 1-1 (why?)

ind hyp: $|A'| \leq |B'|$.

$\Rightarrow \ |A| = |A'| + 1 \leq |B'| + 1 = |B|$ $\boxtimes$

THM: $f: A \xrightarrow{\text{onto}} B \quad (|A|, |B| < \infty)$

$\Rightarrow \quad |A| \geq |B|$

proof: ind. on $|B|$.

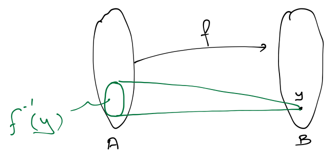
basis: $|B| = 0$. clearly $|A| \geq 0$.

(what is $f: A \to \emptyset$?)

hyp: THM holds for $|B| = n$

step: prove THM for $|B| = n+1$

pick $y \in B$



preimage! f not invertible!

$f^{-1}(y) \triangleq \{x \in A \mid f(x) = y\}$

$A' = A \setminus f^{-1}(y)$

$B' = B \setminus \{y\}$

$g \triangleq$ restriction of f to domain A'.

$g: A' \to B'$ (why?)

g is onto (why?)

f onto $\Rightarrow f^{-1}(y) \neq \emptyset \Rightarrow |A'| \leq |A| - 1$ (*)

hence:

$|A| \underset{(*)}{\geq} |A'| + 1 \underset{\text{ind. hyp}}{\geq} |B'| + 1 = |B|$ $\boxtimes$

CONTRA-POSITIVE

$P \Rightarrow Q \iff \text{not}(Q) \Rightarrow \text{not}(P)$

(proof by contradiction)

(P) $\exists f: A \xrightarrow{1-1} B$

(Q) $|A| \leq |B|$

(not (Q)) $|A| > |B|$

(not (P)) $\forall f: A \to B$ is not 1-1

Pigeonhole Principle: $\text{not}(Q) \Rightarrow \text{not}(P)$