Digital Logic Design: a rigorous approach ©

Chapter 4: Directed Graphs

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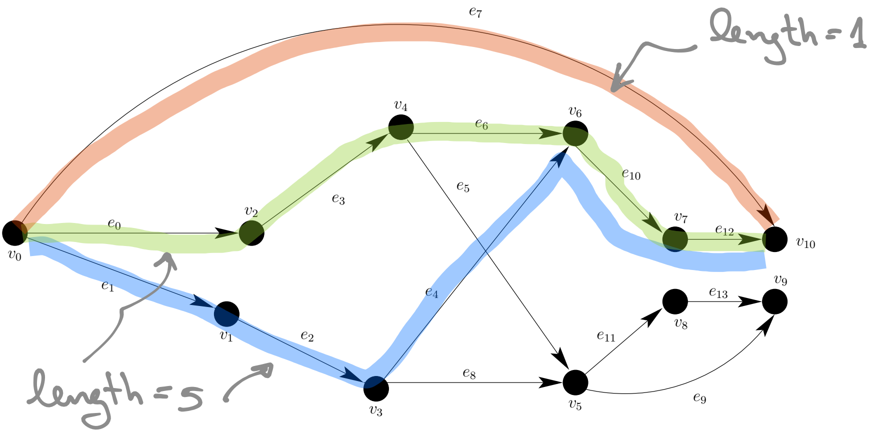
March 18, 2020

Book Homepage:
http://www.eng.tau.ac.il/~guy/Even-Medina

paths ending in V_{10}



longest paths

We denote the length of a path Γ by $|\Gamma|$.

Definition

A path Γ that ends in vertex v is a longest path ending in v if $|\Gamma'| \leq |\Gamma|$ for every path Γ' that ends in v.

Note: there may be multiple longest paths ending in v (hence "a longest path" rather than "the longest path").

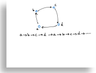
Definition

A path Γ is a longest path in G if $|\Gamma'| \leq |\Gamma|$, for every path Γ' in G.

Question

Does a longest path always exist in a directed graph?

longest paths in DAGs

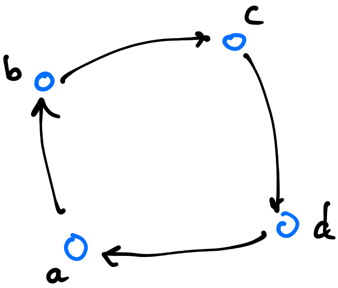


If a directed graph has a cycle, then there does not exist a longest path. Indeed, one could walk around the cycle forever. However, longest paths do exist in DAGs.

Lemma

If $G = (V, E)$ is a DAG, then there exists a longest path that ends in v, for every v. In addition, there exists a longest path in G.

Proof: The length of every path in a DAG is at most $|V| - 1$. [Or, every path is simple, hence, the number of paths is finite.]



$$a \to b \to c \to d \to a \to b \to c \to d \to \cdots$$

computing longest paths: specification

Goal: compute, for every v in a DAG, a longest path that ends in v. We begin with the simpler task of computing the length of a longest path.

Specification

Algorithm longest-path is specified as follows.

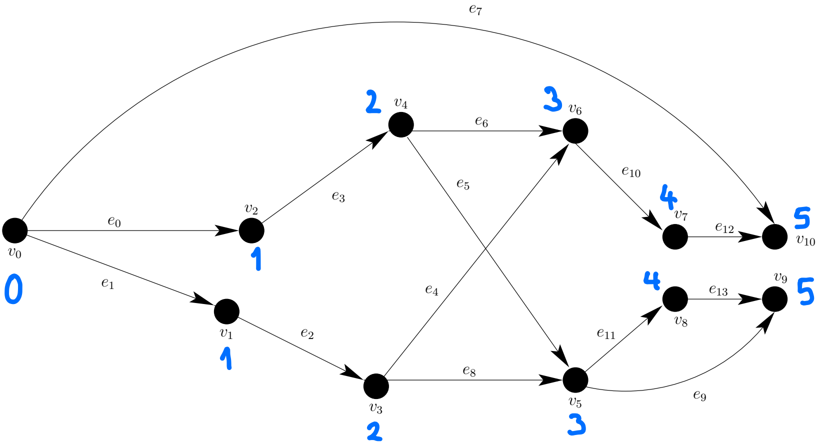
input: A DAG $G = (V, E)$.

output: A delay function $d : V \to \mathbb{N}$.

functionality: For every vertex $v \in V$: $d(v)$ equals the length of a longest path that ends in v.

Application: Model circuits by DAGs. Assume all gates complete their computation in one unit of time. The delay of the output of a gate v equals $d(v)$

example: delay function



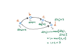
algorithm: longest path lengths

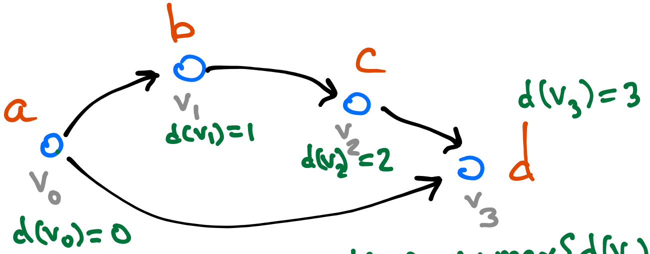
Algorithm 2 longest-path-lengths(V, E) - An algorithm for computing the lengths of longest paths in a DAG. Returns a delay function $d(v)$.

1. topological sort: $(v_0, \ldots, v_{n-1}) \leftarrow TS(V, E)$.
2. For $j = 0$ to $(n-1)$ do
    1. If v_j is a source then $d(v_j) \leftarrow 0$.
    2. Else

$$d(v_j) = 1 + \max\left\{ d(v_i) \mid i < j \text{ and } (v_i, v_j) \in E \right\}.$$

One could design a "single pass" algorithm; the two pass algorithm is easier to prove.





$$d(v_3) = 1 + \max\{d(v_2),\ d(v_0)\}$$
$$= 1 + \max\{2, 0\}$$
$$= 1 + 2 = 3$$

Let

$$d(v) \triangleq \text{output of algorithm}$$
$$\delta(v) \triangleq \text{the length of a longest path that ends in } v$$

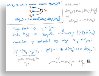
Theorem

Algorithm correct: $\forall j : d(v_j) = \delta(v_j)$.

Proof: Complete induction on j. Basis for sources easy.

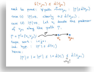
ind. hyp.: $\forall i \leq j : d(v_i) = \delta(v_i)$

step: prove that $d(v_{j+1}) = \delta(v_{j+1})$



We prove now that

1. $\delta(v_{j+1}) \geq d(v_{j+1})$, namely, there exists a path Γ that ends in v_{j+1} such that $|\Gamma| \geq d(v_{j+1})$.
2. $\delta(v_{j+1}) \leq d(v_{j+1})$, namely, for every path Γ that ends in v_{j+1} we have $|\Gamma| \leq d(v_{j+1})$.

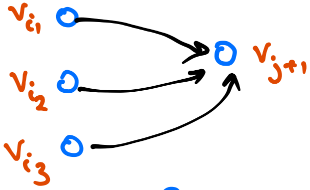


length of longest path ending in v_{j+1} $\rightarrow$ $\delta(v_{j+1}) \geq d(v_{j+1})$ $\leftarrow$ output of alg. longest-path

need to show $\exists$ path $\overset{\Gamma}{\rightsquigarrow} v_{j+1}$: $|\Gamma| \geq d(v_{j+1})$

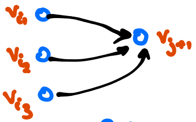
cases: (1) v_{j+1} is a source. easy.

(2) v_{j+1} is not a source. assume $\deg_{in}(v_{j+1}) = 3$



$$d(v_{j+1}) = 1 + \max\{d(v_{i_1}), d(v_{i_2}), d(v_{i_3})\}$$

(2) v_{j+1} is not a source. assume $\deg_{in}(v_{j+1}) = 3$



$$d(v_{j+1}) = 1 + \max\{d(v_{i_1}), d(v_{i_2}), d(v_{i_3})\}$$

assume:

$$d(v_{i_2}) \geq \max\{d(v_{i_1}), d(v_{i_3})\}$$

so:

$$d(v_{j+1}) = 1 + d(v_{i_2})$$

Topo. Sort $\Rightarrow$ $i_2 < j+1$.

ind. hyp. $\Rightarrow$ $\exists$ path $\xrightarrow{\Gamma'} v_{i_2}$: $|\Gamma'| \geq d(v_{i_2})$

consider Γ' extended by edge $v_{i_2} \to v_{j+1}$:

$$\underbrace{|\Gamma' \circ (v_{i_2}, v_{j+1})|}_{\text{path ending in } v_{j+1}} = |\Gamma'| + 1 \geq d(v_{i_2}) + 1 = d(v_{j+1})$$

$$\xrightarrow{\Gamma'} v_{i_2} \longrightarrow v_{j+1} \quad \boxtimes$$

$$\delta(v_{j+1}) \leq d(v_{j+1})$$

need to prove: $\forall$ path $\overset{\Gamma}{\rightsquigarrow} v_{j+1}$: $|\Gamma| \leq d(v_{j+1})$

case (1) $|\Gamma| = 0$. clearly $0 \leq d(v_{j+1})$.

case (2) $|\Gamma| > 0$. Let v_i denote the predessor of v_{j+1} along the path Γ.

$\Gamma = \Gamma' \circ (v_i, v_{j+1})$: $\overset{\Gamma'}{\rightsquigarrow} v_i \rightarrow v_{j+1}$

topo. sort : $i < j+1$.

ind. hyp. : $|\Gamma'| \leq d(v_i)$

hence:

$$|\Gamma| = 1 + |\Gamma'| \leq 1 + d(v_i) \leq d(v_{j+1})$$

why?

Q : how can we actually find a longest path ?

longest-path just tells us its length.

can we find a longest path ?