Digital Logic Design: a rigorous approach ©

Chapter 4: Directed Graphs

Routed Trees

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In the following definition we consider a directed acyclic graph $G = (V, E)$ with a single sink called the root.

Definition

A DAG $G = (V, E)$ is a rooted tree if it satisfies the following conditions:

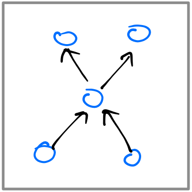
1. There is a single sink in G.
2. For every vertex in V that is not a sink, the out-degree equals one.

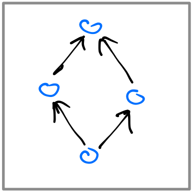
The single sink in rooted tree G is called the root, and we denote the root of G by $r(G)$.

Rooted Trees

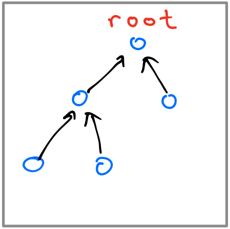
directed acyclic graph $G = (V, E)$ s.t.

1) single sink
2) $\forall v: \deg_{out}(v) \le 1$





not rooted trees



rooted tree

paths to the root

Definition

A DAG $G = (V, E)$ is a rooted tree if it satisfies the following conditions:

1. There is a single sink in G.
2. For every vertex in V that is not a sink, the out-degree equals one.

Theorem

In a rooted tree there is a unique path from every vertex to the root.

$G=(V,E)$ rooted tree $\Rightarrow \forall v \ \exists!$ path $v \overset{\text{path}}{\rightsquigarrow}$ root

proof by ind. on $|V|$.

<u>base</u>: $|V|=1$, trivial.

<u>hyp</u>: holds if $|V|=n$.

<u>step</u>: prove for $|V|=n+1$.

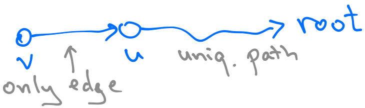
G DAG $\Rightarrow \exists$ source v

consider $G'=(V',E')$ where $\begin{cases} V' \triangleq V \setminus \{v\} \\ E' \triangleq E \setminus E_v \end{cases}$

G' is a rooted tree: $\deg_{out}(u)$ is unchanged

ind. hyp on G': $\forall u \in V' \ \exists!$ path $u \overset{\text{path}}{\rightsquigarrow}$ root

what about v? $\deg_{out}(v)=1 \Rightarrow \exists! u : (v,u) \in E$



2nd proof: 1) $\exists$ path to root

2) unique path to root

$\exists$ path to root:

pick $v \in V$. build path recursively as follows:

$v_0 \leftarrow v$

if v_i sink stop.

if $v_i \neq$ sink, $\exists u : (v_i, u) \in E$.

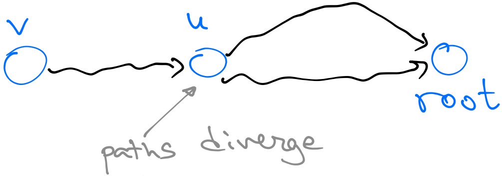
set $v_{i+1} \leftarrow u$.

since $|path| < \infty$, alg. must terminate.

sink is unique, path reaches the root.

2) unique path to root

if $\exists\, 2$ paths : $v \rightsquigarrow$ root



$\Rightarrow \deg_{out}(u) \geq 2$

$\Rightarrow$ contra. to G is a rooted tree.

$\boxtimes$

composition & decomposition of rooted trees

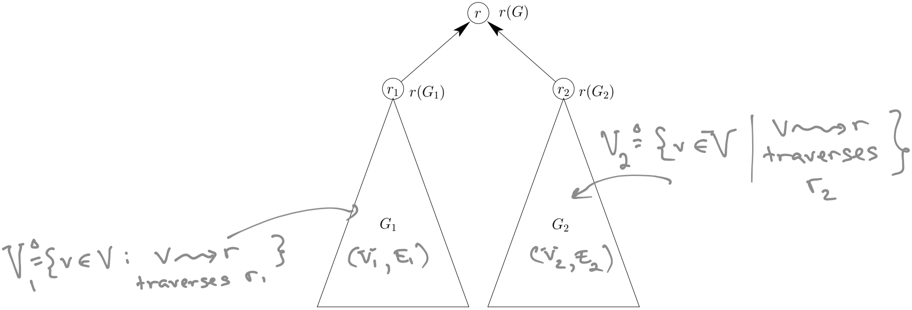


Figure: A decomposition of a rooted tree G into two rooted trees G_1 and G_2.

Terminology

- each the rooted tree $G_i = (V_i, E_i)$ is called a tree hanging from $r(G)$.
- Leaf : a source node.
- interior vertex : a vertex that is not a leaf.
- parent : if $u \longrightarrow v$, then v is the parent of u.
- Typically maximum in-degree$= 2$.

- The rooted trees hanging from $r(G)$ are ordered. Important in parse trees.
- Arcs are oriented from the leaves towards the root. Useful for modeling circuits:
    - leaves = inputs
    - root = output of the circuit.