Digital Logic Design: a rigorous approach ©

Chapter 3: Sequences and Series

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Book Homepage:
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Sequences

Definition

An infinite sequence is a function f whose domain is $\mathbb{N}$ or $\mathbb{N}^+$.

Instead of denoting a sequences by a function $f : \mathbb{N} \to \mathbb{R}$, one usually writes $\{f(0), f(1), \ldots\}$, $\{f(n)\}_{n=0}^{\infty}$ or $\{f_n\}_{n=0}^{\infty}$. Sometimes sequences are only defined for $n \geq 1$.

Example

- $\{g(n)\}_{n=0}^{\infty}$ - the Fibonacci sequence
- $\{d_n\}_{n=0}^{\infty}$ where d_n is the nth digit of $\pi \approx 3.1415926$. Namely, $d(0) = 3, d(1) = 1, d(2) = 4, \ldots$

finite sequences are n-tuples

Definition

A prefix of $\mathbb{N}$ is a set $\{i \in \mathbb{N} \mid i \leq n\}$, for some $n \in \mathbb{N}$.

One could similarly consider prefixes of $\mathbb{N}^+$.

Definition

A finite sequence is a function f whose domain is a prefix of $\mathbb{N}$ or $\mathbb{N}^+$.

Equivalent representations:

- A sequence $\{f(i)\}_{i=0}^{n-1}$
- An n-tuple $(f(0), f(1), \ldots, f(n-1)\}$.

important sequences

- arithmetic sequences
- geometric sequences
- the harmonic sequence

The simplest sequence is the sequence $(0, 1, 2, 3 \dots)$ defined by $f(n) = n$.
An arithmetic sequence is specified by two parameters:

- a_0 - the first element in the sequence
- d- the difference between successive elements.

Example

For the sequence $f(n) = n$:

- $a_0 = 0$
- $d = 1$

Definition

The arithmetic sequence $\{a_n\}_{n=0}^{\infty}$ specified by the parameters a_0 and d is defined by

$$a_n \triangleq a_0 + n \cdot d.$$

Claim

$\{a_n\}_{n=0}^{\infty}$ is an arithmetic sequence iff $\exists d \forall n : a_{n+1} - a_n = d$.

Equivalent definition by recursion:

Definition

- Base case: for $n = 0$, a_0 is given.
- The reduction rule: $a_{n+1} = a_n + d$.

arithmetic sequence - examples

1. The sequence of even numbers $\{e_n\}_{n=0}^{\infty}$ is defined by

$$e_n \triangleq 2n \,.$$

   The sequence $\{e_n\}_{n=0}^{\infty}$ is an arithmetic sequence since $e_{n+1} - e_n = 2$, thus the difference between consecutive elements is constant, as required.

2. The sequence of odd numbers $\{\omega_n\}_{n=0}^{\infty}$ is defined by

$$\omega_n \triangleq 2n + 1 \,.$$

   The sequence $\{\omega_n\}_{n=0}^{\infty}$ is also an arithmetic sequence since $\omega(n+1) - \omega(n) = 2$.

3. If $\{a_n\}_{n=0}^{\infty}$ is an arithmetic sequence with a difference d, then $\{b_n\}_{n=0}^{\infty}$ defined by $b_n = a_{2n}$ is also an arithmetic sequence. Indeed, $b_{n+1} - b_n = a_{2n+2} - a_{2n} = 2d$.

The simplest example of a geometric sequence is the sequence of powers of 2: $(1, 2, 4, 8, \ldots)$. In general, a geometric sequence is specified by two parameters: b_0 - the first element and q - the ratio or quotient between successive elements.

Definition

The geometric sequence $\{b_n\}_{n=0}^{\infty}$ specified by the parameters b_0 and q is defined by

$$b_n \triangleq b_0 \cdot q^n .$$

Claim

$\{b_n\}_{n=0}^{\infty}$ is a geometric sequence iff $\exists q \forall n : b_{n+1}/b_n = q$.

Definition by recursion: The first element is simply b_0. The recursion step is $b_{n+1} = q \cdot b_n$.

geometric sequence - examples

1. The sequence of powers of 3 $\{3^n\}_{n=0}^{\infty}$ is a geometric sequence.
2. If $\{b_n\}_{n=0}^{\infty}$ is a geometric sequence with a quotient q, then $\{c_n\}_{n=0}^{\infty}$ defined by $c_n = b_{2n}$ is also a geometric sequence. Indeed, $\frac{c_{n+1}}{c_n} = \frac{b_{2n+2}}{b_{2n}} = \frac{b_{2n+2}}{b_{2n+1}} \cdot \frac{b_{2n+1}}{b_{2n}} = q^2$.
3. If $\{b_n\}_{n=0}^{\infty}$ is a geometric sequence with a quotient q and $b_n > 0$, then the sequence $\{a_n\}_{n=0}^{\infty}$ defined by $a_n \triangleq \log b_n$ is an arithmetic sequence. Indeed,
   $\log b_n = \log(b_0 \cdot q^n) = \log(b_0) + n \log q$.
4. If $q = 1$ then the sequence $b_n = a_0 \cdot q^n$ is constant.

Definition

The harmonic sequence $\{c_n\}_{n=1}^{\infty}$ is defined by $c_n \triangleq \frac{1}{n}$, for $n \geq 1$.

Note that the first index in the harmonic sequence is $n = 1$. The harmonic sequence is simply the sequence $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$.

Series

The sum of elements in a sequence is called a series. We are interested in the sum of the first n elements of sequences. Given a sequence $\{a_i\}_i$ we are interested in sums

$$\sum_{i \leq n} a_i = a_0 + a_1 + \cdots + a_n$$

We generalize the formula

$$\sum_{i=0}^{n} i = \frac{1}{2} \cdot n(n+1).$$

Theorem

If $a_n \triangleq a_0 + n \cdot d$ *and* $S_n \triangleq \sum_{i=0}^{n} a_i$*,*
Then

$$S_n = a_0 \cdot (n+1) + d \cdot \frac{n \cdot (n+1)}{2}.$$

Proof: by induction (standard) or by reduction.

Question

How fast does S_n grow?

Theorem

Assume that $q \neq 1$. *Let*

$$b_n \triangleq b_0 \cdot q^n \text{ and } \qquad S_n \triangleq \sum_{i=0}^{n} b_i .$$

Then,

$$S_n = b_0 \cdot \frac{q^{n+1} - 1}{q - 1}. \tag{1}$$

Example

- $\sum_{i=0}^{n-1} 2^i = 2^n - 1$ (i.e., $b_0 = 1, q = 2$).
- $\sum_{i=1}^{n} 2^{-i} = \frac{1}{2} \cdot \frac{1-2^{-n}}{1-1/2} = 1 - 2^{-n}$ (i.e., $b_0 = \frac{1}{2}, q = \frac{1}{2}$).

Theorem

Let

$$H_n \triangleq \sum_{i=1}^{n} \frac{1}{i}.$$

Then, for every $k \in \mathbb{N}$

$$1 + \frac{k}{2} \leq H_{2^k} \leq 1 + k. \tag{2}$$

The theorem is useful because it tells us that H_n grows logarithmically in n. In particular, H_n tends to infinity as n grows. Compare with $\int_1^n \frac{1}{x} dx$.