Digital Logic Design: a rigorous approach ©

Chapter 14: Shifters

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Book Homepage:
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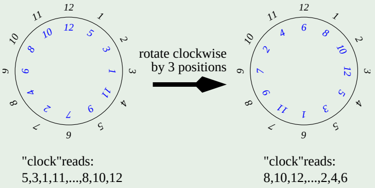
Preliminary questions:

1. Which types of shifts are you familiar with in your favorite programming language? What is the differences between these shifts? Why do we need different types of shifts?
2. How are these shifts executed in a microprocessor?
3. Should shifters be considered to be combinational circuits? After all, they simply "move bits around" and do not compute "new bits".

Cyclic Shifters

Example

- assume that we place the bits of $a[1:12]$ on a wheel.
- $a[1]$ is at one o'clock, $a[2]$ is at two o'clock, etc.
- rotate the wheel, and read the bits in clockwise order starting from one o'clock and ending at twelve o'clock.
- the resulting string is a cyclic shift of $a[1:12]$.



Definition of a Cyclic Shifter

We denote (a mod b) by $\mathrm{mod}(a, b)$.

Definition

The string $b[n-1:0]$ is a cyclic left shift by i positions of the string $a[n-1:0]$ if

$$\forall j: \quad b[j] = a[\mathrm{mod}(j-i, n)].$$

Example

Let $a[3:0] = 0010$. A cyclic left shift by one position of $\vec{a}$ is the string 0100. A cyclic left shift by 3 positions of $\vec{a}$ is the string 0001.

Definition of BARREL-SHIFTER(n)

Definition

A BARREL-SHIFTER(n) is a combinational circuit defined as follows:

Input: $x[n-1:0] \in \{0,1\}^n$ and $sa[k-1:0] \in \{0,1\}^k$ where $k = \lceil \log_2 n \rceil$.

Output: $y[n-1:0] \in \{0,1\}^n$.

Functionality: $\vec{y}$ is a cyclic left shift of $\vec{x}$ by $\langle \vec{sa} \rangle$ positions. Formally,

$$\forall j \in [n-1:0]: \quad y[j] = x[\mathrm{mod}(j - \langle \vec{sa} \rangle, n)].$$

We often refer to the input $\vec{x}$ as the data input and to the input $\vec{sa}$ as the shift amount input. **To simplify the discussion, we assume that n is a power of 2, namely, $n = 2^k$.**

$\textsc{Barrel-Shifter}(n)$ Implementation

We break the task of designing a barrel shifter into smaller sub-tasks of shifting by powers of two. We define this sub-task formally as follows.
A $\textsc{cls}(n, 2^i)$ is a combinational circuit that implements a cyclic left shift by zero or 2^i positions depending on the value of its select input.

Definition

A $\textsc{cls}(n, i)$ is a combinational circuit defined as follows:

Input: $x[n-1:0]$ and $s \in \{0, 1\}$.

Output: $y[n-1:0]$.

Functionality:

$$\forall j \in [n-1:0]: \quad y[j] = x[\mathrm{mod}(j - s \cdot i, n)].$$

Subtask: $\textsc{cls}(n, i)$ Implementation

A $\textsc{cls}(n, i)$ is quite simple to implement since:

- $y[j]$ is either $x[j]$ or $x[\mathrm{mod}(j - i, n)]$.
- So all one needs is a $\textsc{mux}$-gate to select between $x[j]$ or $x[\mathrm{mod}(j - i, n)]$.
- The selection is based on the value of s.
- It follows that the delay of $\textsc{cls}(n, i)$ is the delay of a $\textsc{mux}$, and the cost is n times the cost of a $\textsc{mux}$.

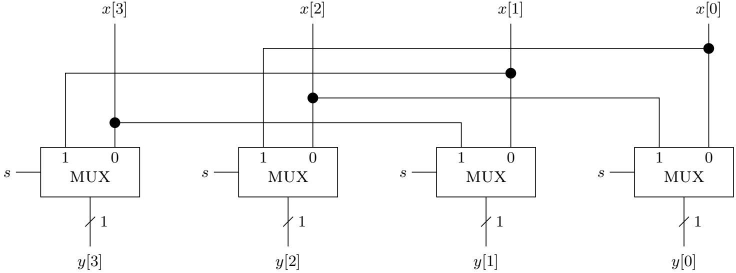


Figure: A row of multiplexers implement a $\textsc{cls}(4, 2)$.

Back to BARREL-SHIFTER(n)

- The design of a BARREL-SHIFTER(n) is based on CLS($n, 2^i$) shifters.
- The implementation is based on k levels of CLS($n, 2^i$), for $i \in [k-1:0]$.
- The ith level is controlled by $sa[i]$.

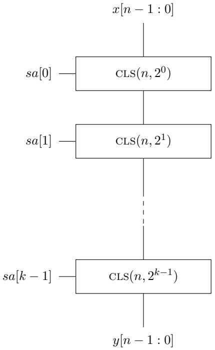


Figure: A BARREL-SHIFTER(n) built of k levels of CLS($n, 2^i$) ($n = 2^k$).

Correctness

Observation

For every $x, q \in \mathbb{Z}$*,*

$$\mathrm{mod}(x, n) = \mathrm{mod}(x + qn, n).$$

Observation

If $\alpha = \mathrm{mod}(a, n)$ *and* $\beta = \mathrm{mod}(b, n)$*, then*

$$\mathrm{mod}(a - b, n) = \mathrm{mod}(\alpha - \beta, n)\ .$$

Claim

The barrel shifter design depicted in the previous slide is correct.

Proof.

Prove by induction on i, that output of $\textsc{cls}(n, 2^i)$ equals the cyclic left shift of x by $\langle sa[i:0] \rangle$. $\square$

Cost & Delay

Claim

The cost and delay of $\textsc{barrel-shifter}(n)$ *satisfy:*

$$c(\textsc{barrel-shifter}(n)) = n \log_2 n \cdot c(\textsc{mux})$$
$$d(\textsc{barrel-shifter}(n)) = \log_2 n \cdot d(\textsc{mux}).$$

Proof.

Follows from the fact that the design consists of $\log_2 n$ levels of $\textsc{cls}(n, 2^i)$ shifters. $\square$

The cone of the Barrel Shifter

Consider the output $y[0]$ of $\textsc{barrel-shifter}(n)$.

Claim

The cone of the Boolean function implemented by the output $y[0]$ contains at least n elements.

Corollary

The delay of $\textsc{barrel-shifter}(n)$ is asymptotically optimal.

Theorem (Pippenger and Yao 1982)

The cost of every cyclic shifter is $\Omega(n \log n)$. Hence, the cost of $\textsc{barrel-shifter}(n)$ is asymptotically optimal.

Logical Shift

Definition

The binary string $y[n-1:0]$ is a logical left shift by ℓ positions of the binary string $x[n-1:0]$ if

$$y[i] \triangleq \begin{cases} 0 & \text{if } i < \ell \\ x[i-\ell] & \text{if } \ell \leq i < n. \end{cases}$$

Example

$y[3:0] = 0100$ is a logical left shift of $x[3:0] = 1001$ by $\ell = 2$ positions. When we apply a logical left shift to $x[n-1:0]$ by ℓ positions, we obtain the string $x[n-1-\ell:0] \circ 0^{\ell}$.

Fast multiplication

In binary representation, logical shifting to the left by s positions corresponds to multiplying by 2^s followed by modulo 2^n.

Logical Shifters (cont.)

Definition

The binary string $y[n-1:0]$ is a logical right shift by ℓ positions of the binary string $x[n-1:0]$ if

$$y[i] \triangleq \begin{cases} 0 & \text{if } i \geq n-\ell \\ x[i+\ell] & \text{if } 0 \leq i < n-\ell. \end{cases}$$

Example

$y[3:0] = 0010$ is a logical right shift of $x[3:0] = 1001$ by $\ell = 2$ positions. When we apply a logical right shift to $x[n-1:0]$ by ℓ positions, we obtain the string $0^{\ell} \circ x[n-1:\ell]$.

Fast division

In binary representation, logical shifting to the right by s positions corresponds to the integer part of the quotient after division by 2^s.

Notation.

- Let $\textsc{lls}(\vec{x}, i)$ denote the logical left shift of $\vec{x}$ by i positions.
- Let $\textsc{lrs}(\vec{x}, i)$ denote the logical right shift of $\vec{x}$ by i positions.

A bi-directional logical shifter

Definition

A $\textsc{l-shift}(n)$ is a combinational circuit defined as follows:

Input:

- $x[n-1:0] \in \{0,1\}^n$,
- $sa[k-1:0] \in \{0,1\}^k$, where $k = \lceil \log_2 n \rceil$, and
- $\ell \in \{0,1\}$.

Output: $y[n-1:0] \in \{0,1\}^n$.

Functionality: The output $\vec{y}$ satisfies

$$\vec{y} \triangleq \begin{cases} \textsc{lls}(\vec{x}, \langle \vec{sa} \rangle) & \text{if } \ell = 1, \\ \textsc{lrs}(\vec{x}, \langle \vec{sa} \rangle) & \text{if } \ell = 0. \end{cases}$$

Question

Design a bi-directional shifter using a left shifter and a right shifter (and select the answer based on ℓ).

A bi-directional logical shifter (cont.)

Example

- let $x[3:0] = 0010$.
- If $sa[1:0] = 10$ and $\ell = 1$, then $\textsc{l-shift}(4)$ outputs $y[3:0] = 1000$.
- If $\ell = 0$, then the output equals $y[3:0] = 0000$.

Implementation

As in the case of cyclic shifters, we break the task of designing a logical shifter into sub-tasks of logical shifts by powers of two.

Definition

An $\text{LBS}(n, i)$ is a combinational circuit defined as follows:

Input: $x[n-1:0]$ and $s, \ell \in \{0, 1\}$.

Output: $y[n-1:0]$.

Functionality: The output $\vec{y}$ satisfies

$$\vec{y} \triangleq \begin{cases} \vec{x} & \text{if } s = 0, \\ \text{LLS}(\vec{x}, i) & \text{if } s = 1 \text{ and } \ell = 1, \\ \text{LRS}(\vec{x}, i) & \text{if } s = 1 \text{ and } \ell = 0. \end{cases}$$

The role of the input s in is to determine if a shift (in either direction) takes place at all. If $s = 0$, then $y[j] = x[j]$, and no shift takes place. If $s = 1$, then the direction of the shift is determined by ℓ.

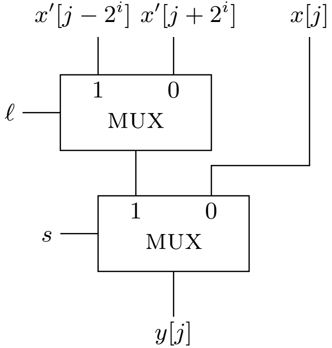


Figure: A bit-slice of an implementation of $\mathrm{LBS}(n, 2^i)$.

question

Design a bi-directional logical shifter $\mathrm{L\text{-}SHIFT}(n)$ by cascading $\mathrm{LBS}(n, 2^i)$ shifters.

Reduction of right shift to left shift

Definition

Let *rev* : $\{0,1\}^* \rightarrow \{0,1\}^*$ denote the function that reverses strings. Formally:

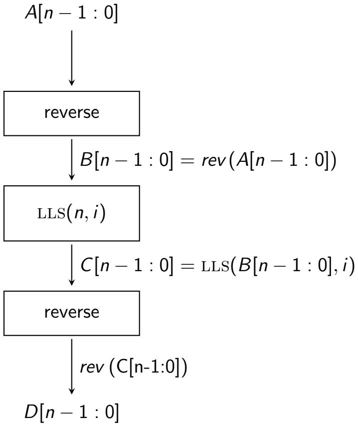
$$rev(A_{n-1}, \ldots, A_1, A_0) = (A_0, A_1, \ldots, A_n).$$

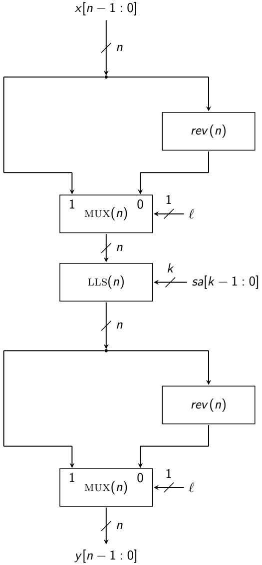
Reversing a string can be implemented with zero cost and zero delay. All one needs to do is connect input $A[i]$ to the output $B[n-i]$.

Reduction of right shift to left shift (cont.)

Claim

$\text{LRS}(\vec{x}, i) = rev(\text{LLS}(rev(\vec{x}), i)).$




$x[n-1:0]$
n
$rev(n)$
1
$\text{MUX}(n)$
0
1
ℓ
n
$\text{LLS}(n)$
k
$sa[k-1:0]$
n
$rev(n)$
1
$\text{MUX}(n)$
0
1
ℓ
n
$y[n-1:0]$

Arithmetic Shifters

Arithmetic shifters are used for shifting binary strings that represent signed integers in two's complement representation. **Since left shifting is the same in logical shifting and in arithmetic shifting, we discuss only right shifting** (i.e., division by a power of 2).

Definition

The binary string $y[n-1:0]$ is an arithmetic right shift by ℓ positions of the binary string $x[n-1:0]$ if the following holds:

$$y[i] \triangleq \begin{cases} x[n-1] & \text{if } i \geq n-\ell \\ x[i+\ell] & \text{if } 0 \leq i < n-\ell. \end{cases}$$

Arithmetic Shifters (cont.)

Example

- $y[3:0] = 0010$ is an arithmetic shift of $x[3:0] = 0101$ by $\ell = -1$ positions.
- On the other hand, $y[3:0] = 1110$ is an arithmetic shift of $x[3:0] = 1001$ by $\ell = -2$ positions.
- When we apply an arithmetic shift by $\ell < 0$ positions to $x[n-1:0]$, we obtain the string $x[n-1]^{\ell} \circ x[n-1:\ell]$.

Notation.
Let $\text{ARS}(\vec{x}, i)$ denote the arithmetic right shift of $\vec{x}$ by i positions.

An arithmetic right shifter

Definition

An $\textsc{arith-shift}(n)$ is a combinational circuit defined as follows:

Input: $x[n-1:0] \in \{0,1\}^n$ and $sa[k-1:0] \in \{0,1\}^k$, where $k = \lceil \log_2 n \rceil$.

Output: $y[n-1:0] \in \{0,1\}^n$.

Functionality: The output $\vec{y}$ is a (sign-extended) arithmetic right shift of $\vec{x}$ by $\langle \vec{sa} \rangle$ positions. Formally,

$$y[n-1:0] \triangleq \textsc{ars}(x[n-1:0], \langle \vec{sa} \rangle).$$

Example

Let $x[3:0] = 1001$. If $sa[1:0] = 10$, then $\textsc{arith-shift}(4)$ outputs $y[3:0] = 1110$.

question

Design an arithmetic right shifter $\textsc{arith-shift}(n)$.