Digital Logic Design: a rigorous approach ©

Chapter 12: Trees

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Preliminary questions:

1. Which Boolean functions are suited for implementation by tree-like combinational circuits?
2. In what sense are tree-like implementations optimal?

Reminder: Binary Boolean Functions

Definition

A binary Boolean function is a function $f : \{0,1\}^2 \to \{0,1\}$.

A binary function is often denoted by a dyadic operator, say $*$. So instead of writing $f(a, b)$, we write $a * b$.

Reminder: Associative Boolean functions

Definition

A binary Boolean function $* : \{0,1\}^2 \to \{0,1\}$ is associative if

$$(x_1 * x_2) * x_3 = x_1 * (x_2 * x_3) \ ,$$

for every $x_1, x_2, x_3 \in \{0,1\}$.

One may omit parenthesis: $x_1 * x_2 * x_3$ is well defined.
Consider the function $f_n : \{0,1\}^n \to \{0,1\}$ defined by

$$f_n(x_1, \ldots, x_n) \triangleq x_1 * \cdots * x_n$$

Extension of associative function

Definition

Let $f : \{0,1\}^2 \to \{0,1\}$ denote a Boolean function. The function $f_n : \{0,1\}^n \to \{0,1\}$, for $n \geq 1$, is defined recursively as follows.

1. If $n = 1$, then $f_1(x) = x$.
2. If $n = 2$, then $f_2 = f$.
3. If $n > 2$, then f_n is defined based on f_{n-1} as follows:

$$f_n(x_1, x_2, \ldots x_n) \triangleq f(f_{n-1}(x_1, \ldots, x_{n-1}), x_n).$$

Claim

If $f : \{0,1\}^2 \to \{0,1\}$ is an associative Boolean function, then

$$f_n(x_1, x_2, \ldots x_n) = f(f_{n-k}(x_1, \ldots, x_{n-k}), f_k(x_{n-k+1}, \ldots, x_n)),$$

for every $n \geq 2$ and $k \in [1, n-1]$.

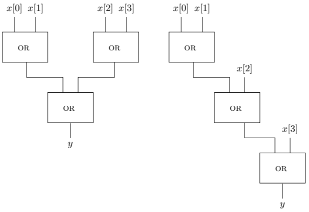
Trees of associative Boolean gates

To simplify the presentation, consider the Boolean function OR_n.

Definition

A combinational circuit $H = (V, E, \pi)$ that satisfies the following conditions is called an OR-tree(n).

1. The graph $DG(H)$ is a rooted tree with n sources.
2. Each vertex v in V that is not a source or a sink is labeled $\pi(v) = \mathrm{OR}$.
3. The set of labels of leaves of H is $\{x_0, \dots, x_{n-1}\}$.



Correctness of $\textsc{or}$-tree(n)

Definition

A combinational circuit $H = (V, E, \pi)$ that satisfies the following conditions is called an $\textsc{or}$-tree(n).

1. *Topology.* The graph $DG(H)$ is a rooted tree with n sources.
2. Each vertex v in V that is not a source or a sink is labeled $\pi(v) = \textsc{or}$.
3. The set of labels of leaves of H is $\{x_0, \ldots, x_{n-1}\}$.

Claim

Every $\textsc{or}$*-tree*(n) *implements the Boolean function* $\textsc{or}_n$.

Relation to Boolean Formulas

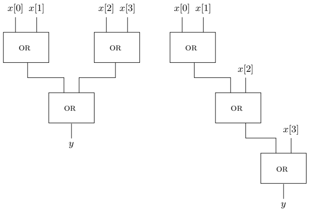
Definition

A Boolean formula φ is an $\textsc{or}(n)$ formula if it satisfies three conditions: (i) it is over the variables $X_0, \ldots, X_{n-1}$, (ii) every variable X_i appears exactly once in φ, and (iii) the only connective in φ is the $\textsc{or}$ connective.

Claim

A Boolean circuit C is an $\textsc{or}(n)$-tree if and only if its graph (without the input/output gates) is a parse tree of an $\textsc{or}(n)$-formula.

Cost of OR-tree(n)



Claim

The cost of every OR*-tree*(n) *is* $(n-1) \cdot c(\text{OR})$.

Lemma

Let $G = (V, E)$ *denote a rooted tree in which the in-degree of each vertex is at most two. Then*

$$|\{v \in V \mid deg_{in}(v) = 2\}| \;=\; |\{v \in V \mid deg_{in}(v) = 0\}| - 1.$$

Depth of tree

delay of an OR tree $=$ number of OR-gates along the longest path from an input to an output.

Definition (depth - nonstandard definition)

The depth of a rooted tree T is the maximum number of vertices with in-degree greater than one in a path in T. We denote the depth of T by $depth(T)$.

Why is this nonstandard?

- Usually, depth is simply the length of the longest path.
- Here we count only vertices with in-degree ≥ 2.
- Why?
  - Input and output gates have zero delay (no computation)
  - Assume inverters are free and have zero delay (we will show that for $\text{OR}(n)$ cost & delay are not reduced even if inverters are free and without delay)

Binary rooted trees

Definition

A rooted tree is a binary tree if the maximum in-degree is two.

A rooted tree is a minimum depth tree if its depth is minimum among all the rooted trees with the same number of leaves.
All binary trees with n leaves have the same cost. But, which have minimum depth?

1. if n that is a power of 2, then there is a unique minimum depth tree, namely, the perfect binary tree with $\log_2 n$ levels.
2. if n is not a power of 2, then there is more than one minimum depth tree... (balanced trees)

Are these minimum depth trees?

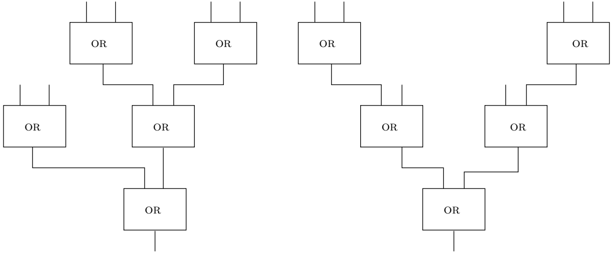


Figure: Two trees with six inputs.

Depth: lower bound

Claim

If T_n is a rooted binary tree with n leaves, then the depth of T_n is at least $\lceil \log_2 n \rceil$.

1. Suffice to prove depth $\geq \log_2 n$.
2. Complete induction on n.

Min Depth: the case $n = 2^k$ (perfect binary trees)

The distance of a vertex v to the root r in a rooted tree is the length of the path from v to r.

Definition

A rooted binary tree is perfect if:

- The in-degree of every non-leaf is 2, and
- All leaves have the same distance to the root.

Note that the depth of a perfect tree equals the distance from the leaves to the root (no vertices with in-degree 1).

Claim

The number of leaves in a perfect tree is 2^k, where k is the distance of the leaves to the root.

Claim

Let n denote the number of leaves in a perfect tree. Then, the distance from every leaf to the root is $\log_2 n$.

We now show that for every n, we can construct a minimum depth tree T_n^* of depth $\lceil \log_2 n \rceil$. In fact, if n is not a power of 2, then there are many such trees.

Balanced partitions

Definition

Two positive integers a, b are a balanced partition of n if:

1. $a + b = n$, and
2. $\max\{\lceil \log_2 a \rceil, \lceil \log_2 b \rceil\} \leq \lceil \log_2 n \rceil - 1$.

Claim

If $n = 2^k - r$, where $0 \leq r < 2^{k-1}$, then the set of balanced partitions is

$$P \triangleq \{(a, b) \mid 2^{k-1} - r \leq a \leq 2^{k-1} \text{ and } b = n - a\}.$$

Construction of a balanced tree

Algorithm 1 Balanced-Tree(n) - a recursive algorithm for constructing a binary tree T_n^* with $n \geq 1$ leaves.

1. The case that $n = 1$ is trivial (an isolated root).
2. If $n \geq 2$, then let a, b be balanced partition of n.
3. Compute trees T_a^* and T_b^*. Connect their roots to a new root to obtain T_n^*.

Definition

A rooted binary tree T_n is a balanced tree if it is a valid output of Algorithm Balanced-Tree(n).

Def: balanced tree

Algorithm 2 Balanced-Tree(n) - a recursive algorithm for constructing a binary tree T_n^* with $n \geq 1$ leaves.

1. The case that $n = 1$ is trivial (an isolated root).
2. If $n \geq 2$, then let a, b be balanced partition of n.
3. Compute trees T_a^* and T_b^*. Connect their roots to a new root to obtain T_n^*.

Claim

The depth of a binary tree T_n^* constructed by Algorithm Balanced-Tree(n) is $\lceil \log_2 n \rceil$.

Corollary

The propagation delay of a balanced OR*-tree(n) is $\lceil \log_2 n \rceil \cdot t_{pd}(\text{OR})$.*

Optimality of trees

Goals: prove optimality of a balanced $\textsc{or}$-tree(n).

Theorem

Let C_n denote a combinational circuit that implements $\textsc{or}_n$. Then,

$$c(C_n) \geq n - 1.$$

Theorem

Let C_n denote a combinational circuit that implements $\textsc{or}_n$. Let k denote the maximum fan-in of a gate in C_n. Then

$$t_{pd}(C_n) \geq \lceil \log_k n \rceil .$$

Flipping bits

Definition

Let $\mathit{flip}_i : \{0,1\}^n \rightarrow \{0,1\}^n$ be the Boolean function defined by $\mathit{flip}_i(\vec{x}) \triangleq \vec{y}$, where

$$y_j \triangleq \begin{cases} x_j & \text{if } j \neq i \\ \text{NOT}(x_j) & \text{if } i = j. \end{cases}$$

The cone of a function

Definition (Cone of a Boolean function)

The cone of a Boolean function $f : \{0,1\}^n \to \{0,1\}$ is defined by

$$cone(f) \triangleq \{i : \exists \vec{v} \text{ such that } f(\vec{v}) \neq f(flip_i(\vec{v}))\}$$

Example

$cone(\textsc{xor}) = \{1,2\}$.

We say that f depends on x_i if $i \in cone(f)$.

Example

Consider the following Boolean function:

$$f(\vec{x}) = \begin{cases} 0 & \text{if } \sum_i x_i < 3 \\ 1 & \text{otherwise.} \end{cases}$$

Suppose that one reveals the input bits one by one. As soon as 3 ones are revealed, one can determine the value of $f(\vec{x})$. Nevertheless, the function $f(\vec{x})$ depends on all its inputs, and hence, $cone(f) = \{1, \ldots, n\}$.

Claim

$\text{cone}(f) = \emptyset \iff f$ *is a constant Boolean function.*

Composition of Functions

Claim

If $g(\vec{x}) \triangleq B(f_1(\vec{x}), f_2(\vec{x}))$, *then*

$$\text{cone}(g) \subseteq \text{cone}(f_1) \cup \text{cone}(f_2) \ .$$

Graphical Cone

Definition

Let $G = (V, E)$ denote a DAG. The graphical cone of a vertex $v \in V$ is defined by

$$cone_G(v) \triangleq \{u \in V : deg_{in}(u) = 0 \text{ and } \exists \text{path from } u \text{ to } v\}.$$

In a combinational circuit, every source is an input gate. This means that the graphical cone of v equals the set of input gates from which there exists a path to v.

Claim

Let $H = (V, E, \pi)$ denote a combinational circuit. Let $G = DG(H)$. For every vertex $v \in V$, the following holds:

$$\text{cone}(f_v) \subseteq \text{cone}_G(v) .$$

Namely, if f_v depends on x_i, then the input gate u that feeds the input x_i must be in the graphical cone of v.

"Hidden" Rooted Trees

Claim

Let $G = (V, E)$ denote a DAG. For every $v \in V$, there exist $U \subseteq V$ and $F \subseteq E$ such that:

1. *$T = (U, F)$ is a rooted tree;*
2. *v is the root of T;*
3. $\text{cone}_G(v)$ *equals the set of leaves of (U, F).*

The sets U and F are constructed as follows.

1. Initialize $F = \emptyset$ and $U = \emptyset$.
2. For every source u in $cone_G(v)$ do
   (a) Find a path p_u from u to v.
   (b) Let q_u denote the prefix of p_u, the vertices and edges of which are not contained in U or F.
   (c) Add the edges of q_v to F, and add the vertices of q_v to U.

Lower Bound on Cost

Theorem (Linear Cost Lower Bound Theorem)

Let $H = (V, E, \pi)$ denote a combinational circuit. If the fan-in of every gate in H is at most 2, then

$$c(H) \geq \max_{v \in V} |\text{cone}(f_v)| - 1.$$

Corollary

Let C_n denote a combinational circuit that implements OR_n. Then

$$c(C_n) \geq n - 1.$$

Lower Bound on Delay

Theorem (Logarithmic Delay Lower Bound Theorem)

Let $H = (V, E, \pi)$ denote a combinational circuit. If the fan-in of every gate in H is at most 2, then

$$t_{pd}(H) \geq \max_{v \in V} \log_2 |\text{cone}(f_v)|.$$

Corollary

Let C_n denote a combinational circuit that implements OR_n. Let 2 denote the maximum fan-in of a gate in C_n. Then

$$t_{pd}(C_n) \geq \lceil \log_2 n \rceil .$$

What is the effect of increasing the fan-in on the delay?

Theorem (Logarithmic Delay Lower Bound Theorem)

Let $H = (V, E, \pi)$ denote a combinational circuit. If the fan-in of every gate in H is at most k, then

$$t_{pd}(H) \geq \max_{v \in V} \log_k |\text{cone}(f_v)|.$$

Corollary

Let C_n denote a combinational circuit that implements OR_n. Let k denote the maximum fan-in of a gate in C_n. Then

$$t_{pd}(C_n) \geq \lceil \log_k n \rceil .$$

Summary

- Focus on combinational circuits that have a topology of a tree with identical gates.
- Trees are especially suited for computing associative Boolean functions.
- Defined an OR-tree(n) to be a combinational circuit that implements OR_n using a topology of a tree.
- Proved that OR-tree(n) are asymptotically optimal (cost).
- Balance conditions to obtain good delay.
- General lower bounds based on $cone(f)$.
  - # gates in a combinational circuit implementing a Boolean function f must be at least $|cone(f)| - 1$.
  - the propagation delay of a combinational circuit implementing a Boolean function f is at least $\log_2 |cone(f)|$.